Serre vanishing theorem in the twisted primitive cohomology

NCG Summer School at DLUT

Hao Zhuang

BICMR, PKU

July 18, 2026

Motivation in complex geometry

Suppose that

- X is a compact complex manifold,
- L is a positive holomorphic line bundle over X,
- E is a holomorphic vector bundle over X.

Theorem (Serre, 1955)

There is a $p_0 \in \mathbb{N}$ such that the Dolbeault cohomology

$$H^q(X, L^{\otimes p} \otimes E) = 0$$

for all $p \geqslant p_0$ and $q \geqslant 1$.

Motivation in complex geometry

Symplectic:

- primitive cohomology (Tseng-Yau, 2012),
- filtered cohomology (Tsai-Tseng-Yau, 2016),

Almost complex:

- generalized Dolbeault cohomology (Cirici-Wilson, 2021).

...

Question

Does Serre's vanishing theorem generalize to these "complex-like" cases?

Theorem (Z, 2026)

YES for the twisted Tseng-Yau primitive cohomology of a compact symplectic manifold.

Primitive cohomology

We let

- (M, ω): compact symplectic manifold,
- d: de Rham exterior derivative.

We obtain a cochain complex ($0 \leqslant k \leqslant 1 + \dim M$):

$$\begin{aligned} \mathrm{d}^{\omega} : \Omega^k(M) \oplus \Omega^{k-1}(M) &\to \Omega^{k+1}(M) \oplus \Omega^k(M) \\ (\alpha, \beta) &\mapsto (\mathrm{d}\alpha + \omega \wedge \beta, -\mathrm{d}\beta). \end{aligned}$$

Definition & Theorem (Tseng-Yau, 2012; Tanaka-Tseng, 2018)

The cohomology defined by $\Omega^{\bullet}(M) \oplus \Omega^{\bullet-1}(M)$ and d^{ω} is called the primitive cohomology of (M, ω).

Primitive cohomology

Originally by Tseng-Yau: Via the Lefschetz map.

$\Leftrightarrow$

Current mapping cone model by Tanaka-Tseng.

Example

On the 2-torus $\mathbb{T}^2$:

- Globally defined tangent frame $\{e_1, e_2\}$ and coframe $\{e_1^*, e_2^*\}$.
- Symplectic form $\omega = e_1^* \wedge e_2^*$.

The ranks of the primitive cohomology groups of $(\mathbb{T}^2, \omega)$ are $1, 2, 2, 1$ respectively.

Ways to compute:

- Lefschetz map (Tseng-Yau, 2012; Tsai-Tseng-Yau, 2016).
- Mapping cone Morse theory (Clausen-Tang-Tseng, 2026; Z, 2026).

Twisted primitive cohomology

How about the "twist"?

We need the symplectic flatness:

Definition (Tseng-Zhou, 2022)

For a connection ∇ on a smooth vector bundle $E \to M$, if there is a $\Phi \in \Gamma(\mathrm{End}(E))$ such that

$$\begin{aligned} \nabla^2 + \omega\Phi &= 0, \\ \Phi\nabla &= \nabla\Phi, \end{aligned}$$

then we say (E, ∇, Φ) is symplectically flat.

Twisted primitive cohomology

A simple example:

Example

Trivial line bundle $E = \mathbb{T}^2 \times \mathbb{R}$ over $\mathbb{T}^2$:

- Tangent frame $\{e_1, e_2\}$ and coframe $\{e_1^*, e_2^*\}$ on $\mathbb{T}^2$.
- Symplectic form $\omega = e_1^* \wedge e_2^*$ on $\mathbb{T}^2$.
- Any connection ∇ on E is defined by 1-form $fe_1^* + ge_2^*$ $(f, g \in C^\infty(\mathbb{T}^2))$.

When $e_1^*(g) - e_2^*(f) =$ constant c, we have (E, ∇, c) symplectically flat .

More examples:

Certain critical points of the Yang-Mills functional.

Twisted primitive cohomology

For symplectically flat (E, ∇, Φ), we have the cochain complex $(0 \leqslant k \leqslant 1 + \dim M)$:

$$\begin{aligned} \mathbb{A} : \Omega^k(M, E) \oplus \Omega^{k-1}(M, E) &\to \Omega^{k+1}(M, E) \oplus \Omega^k(M, E) \\ (\alpha, \beta) &\mapsto (\nabla\alpha + \omega \wedge \beta, \Phi\alpha - \mathrm{d}\beta). \end{aligned}$$

Definition & Theorem (Tseng-Zhou, 2022)

The cohomology defined by $\Omega^\bullet(M, E) \oplus \Omega^{\bullet-1}(M, E)$ and $\mathbb{A}$ is called the primitive cohomology of (M, ω) twisted by (E, ∇, Φ).

This twisted cohomology also admits a Lefschetz map model.

Vanishing theorems

Given a symplectically flat (E, ∇, Φ) over (M, ω):

Question

What if Φ is an automorphism?

The answer is a vanishing theorem.

Theorem (Tseng-Zhou, 2022)

When Φ is invertible, the k-th primitive cohomology group of (M, ω) twisted by (E, ∇, Φ) vanishes for all $0 \leqslant k \leqslant 1 + \dim M$.

This cohomology vanishes from $k = 0$, not just from $k = 1$.

Question

What if Φ is just an endomorphism?

Vanishing theorems

(L, ∇', Φ'):

- Symplectically flat (real) line bundle over (M, ω).
- $\Phi' \neq 0$ everywhere.

Naturally induced on $E \otimes L^{\otimes p}$:

- a connection ∇'',
- an endomorphism $\Phi'' \in \Gamma(\mathrm{End}(E \otimes L^{\otimes p}))$.

The answer is a Serre type vanishing theorem:

Theorem (Z, 2026)

There is a $p_0 \in \mathbb{N}$ such that when $p \geqslant p_0$, the k-th primitive cohomology group of (M, ω) twisted by $(E \otimes L^{\otimes p}, \nabla'', \Phi'')$ vanishes for all k.

You can always replace ω by any closed 2-form.

Replacement of the holomorphicity

Why do we care about the symplectic flatness?

Remark

The symplectic flatness is an analogue of the holomorphicity.

Complex case:

Theorem (Atiyah, 1957)

Holomorphic connection exists $\Leftrightarrow$ *Atiyah class vanishes.*

Symplectic case:

Question

Does the symplectic flatness admit an associated characteristic class? For example, an analogue of the Atiyah class?

Primitive characteristic classes

Recall the map

$$\begin{aligned} \mathbb{A} : \Omega^k(M,E) \oplus \Omega^{k-1}(M,E) &\to \Omega^{k+1}(M,E) \oplus \Omega^k(M,E) \\ (\alpha,\beta) &\mapsto (\nabla\alpha + \omega\wedge\beta, \Phi\alpha - \mathrm{d}\beta). \end{aligned}$$

Symplectic flatness $\Leftrightarrow \mathbb{A}^2 = 0$.

This $\mathbb{A}^2$ should be viewed as a "curvature".

Theorem (Z, 2025)

With or without symplectic flatness, tr$(e^{-\mathbb{A}^2})$ *satisfies transgression.*

We call tr$(e^{-\mathbb{A}^2})$ the primitive Chern character.

Primitive characteristic classes

Corollary

Symplectically flat connection $\Rightarrow$

- *1st primitive Chern class vanishes.*
- *primitive Chern character* $= \mathrm{rank}(E)$*.*

It is not the final counterpart of the Atiyah class.

Yang-Mills theory perspective:

Theorem (Tseng-Zhou, 2025)

Symplectically flat connections on principal bundles $\Leftrightarrow$ *conjugacy classes of a certain type of group homomorphisms.*

"holomorphic-like" perspective?

Thank the organizers for the invitation.

Thank you for your attention.

Slides are available at:
https://hzhmp.github.io/events_en.html/ncg_dlut.pdf